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A METHOD OF ESTIMATING THE CRITICAL BUCKLING LOAD FOR STRUCTURAL MEMBERS

By Eugene E. Lundquist
Langley Memorial Aeronautical Laboratory

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SUMMARY

The relations between load on the structure and rotation of a joint can be used to estimate the lowest critical load after the equation for neutral stability has been tested for three assumed critical loads, each of which is less than the lowest critical load.

The solutions of six simple problems are included to illustrate the application of the method of estimating critical loads and to reveal certain characteristics of the method that should be known by the practical engineer using it. Four of these problems are concerned with members that lie in the elastic, or long-column, range. The other two problems are concerned with members that lie in the short-column range.

INTRODUCTION

One of the problems in the design of structures is to make certain that the compression members are stable under the loads to be carried. For structures built with the members joined to each other by frictionless pins, the usual column formulas can be directly applied to the design of the compression members. For structures built with the members continuous at the joints, however, the design of any one member is dependent upon the design of all other members.

Reference 1 shows how the principles of the Cross method of moment distribution can be used to check the stability of structural members under axial load and hence the safety of the design. In this method, the critical load for the system of members is calculated and compared with the applied load. If the critical load is greater than the applied load, the members are stable. If the critical load is less than the applied load, the system is unstable and a larger size for one or more of the compression members must be selected.

One disadvantage of any method of calculating the critical load for a system of structural members under axial load is that, for each type of instability, there is a corresponding critical load. In design, the lowest critical load is the only one of interest. When the stability of a group of structural members is checked, it is therefore the lowest of these critical loads that must be calculated and compared with the applied loads.

Although the two equations for neutral stability given in reference 1 are algebraic in appearance, they are fundamentally transcendental in character with the unknown critical load entering in angles. The method of solution used in reference 1 was to assume several values of the critical load and to test one of the equations for neutral stability. If the first load in the series of assumed loads is made sufficiently small, the lowest assumed load that just satisfies this equation is the critical load desired.

Unless the designer is fortunate in selecting the assumed critical loads, considerable time and labor are required to find the lowest critical load. In order to make the theory of reference 1 more useful in practical calculations, a method of estimating the lowest critical load is presented in this report.

In this paper, as in reference 1, it is assumed that the members lie in a plane and that buckling occurs in this plane. It is further assumed that the joints of the structure are held rigidly in space but are free to rotate under the elastic restraint of the interconnecting members.

DEFINITIONS AND SYMBOLS

The following definitions of stiffness and carry-over factor are the same as those given in references 1 and 2.

Stiffness

If a member is on unyielding supports at each end, the moment at one end necessary to produce a rotation of one-fourth radian of that end is called the "stiffness." The stiffness of a member will depend upon the amount of restraint at the far end. In the derivation of the criterion for stability as given in reference 1, three types of restraint at the far end are considered. The symbols used to designate the stiffness for the different types of restraint are:

S, far end fixed.

S', far end elastically restrained.

S'', far end pinned.

The stiffness of a member computed according to the foregoing definition is one-fourth that computed according to the definition given in references 3 and 4. In the Cross method, the relative stiffness of the members is of importance and not the absolute value. The foregoing definition was selected so that the stiffness of a member of constant cross section with no axial load and fixed at the far end would be $\overline{E}I/L$ instead of $4\overline{E}I/L$.

Carry-Over Factor

If a member is on unyielding supports at each end and a moment is applied at the near end, the ratio of the moment developed at the far end to the moment applied at the near end is called the "carry-over factor." As in the case of stiffness, the carry-over factor will depend upon the degree of restraint at the far end of the member. The symbols used to designate the carry-over factor for the different types of restraint are:

C, far end fixed.

C', far end elastically restrained.

$C'' = 0$, far end pinned.

Sign Convention

The sign convention used in this report is the same as that used in references 1, 2, and 4. A clockwise moment acting on the end of a member is positive. A counterclockwise moment acting on a joint is positive. An external moment applied at a joint is considered to act on the joint. A positive moment acting on the end of a member causes positive rotation of that end.

Symbols

E, modulus of elasticity.

$\overline{E}$, effective modulus of elasticity.

I, moment of inertia of cross section of member about a centroidal axis normal to the plane of bending.

L, length of member.

W, total load on the structure.

P, axial load in member (absolute value).

A, area of cross section.

c, restraint coefficient in the usual column formula.

$\rho = \sqrt{\frac{I}{A}}$, radius of gyration.

$$j = \sqrt{\frac{EI}{P}}$$

$$\frac{L}{j} = \frac{L}{\sqrt{\frac{EI}{P}}}$$

$$\left(\frac{L}{j}\right)_{eff.} = \frac{L}{\sqrt{\frac{\overline{E}I}{P}}}$$

$$\tau = \frac{\overline{E}}{E}$$

METHOD OF ESTIMATING THE CRITICAL LOAD

The method of estimating the lowest critical load is based upon the principles discussed in references 5 and 6 for the analysis of experimental observations in problems of elastic stability. In reference 5, Southwell mentions that the unavoidable imperfections in practical structures prevent the realization of the concept of a critical load at which deflections begin. Instead, the initial deflections present in practical structures steadily increase with load and, according to the usual theory, the deflections become infinite as the critical load is approached.

In references 5 and 6, the relation between load and deflection for problems of elastic stability is also discussed. The more general relation given in reference 6 shows that, if $\frac{y - y_1}{P - P_1}$ is plotted as ordinate against $y - y_1$ as abscissa, the curve obtained when P approaches P_{crit} is essentially a straight line the inverse slope of which is $P_{crit} - P_1$, where

y is deflection at axial load P in a member.

y_1 and P_1, initial values of y and P, respectively.

P_{crit}, lowest critical load.

$$P_1 < P < P_{crit} \tag{1}$$

Thus, if simultaneous readings of load and deflection recorded in a test are plotted as just described beginning with any load P_1 as the initial reading, the value of $P_{crit} - P_1$ is readily obtained. The value of P_{crit} is then given by the relation

$$P_{crit} = (P_{crit} - P_1) + P_1 \tag{2}$$

The relation between load and deflection can also be applied to load and rotation of a joint. In order to use this relation in theoretical calculations, there must be initial rotation of the joints. This rotation is obtained by a fictitious external moment M applied at some joint, after which the load on the structure is imagined to be applied. The effect of the tension in the tension members is such as to reduce the rotations caused by the external moment M; whereas the effect of the compression in the compression members is such as to increase the rotations. As the lowest critical load is approached, the effects of compression completely overshadow the effects of tension with the result that the rotations become infinite.

If the distribution of the total load W on the structure does not change as W increases, then the axial load in each member is proportional to W. Thus, if $\frac{\theta - \theta_1}{W - W_1}$ is plotted as ordinate against $\theta - \theta_1$ as abscissa, the curve obtained when W approaches W_{crit} is essentially a straight line the inverse slope of which is $W_{crit} - W_1$, where

θ is rotation of a joint under the moment M at load W on the structure.

θ_1 and W_1, initial values of θ and W, respectively.

W_{crit}, lowest critical load.

and

$$W_1 < W < W_{crit} \tag{3}$$

Thus, if simultaneous values of load and rotation are plotted as just described beginning with W_1 as the initial load, the value of $W_{crit} - W_1$ is easily obtained. The value of W_{crit} is then given by the equation

$$W_{crit} = (W_{crit} - W_1) + W_1 \tag{4}$$

The procedure to be used in estimating the critical load for a group of structural members is:

1. Assume three values of W that are known to be less than W_{crit}. This condition is satisfied if the values of W are selected so that the axial load in each compression member is less than the strength of that member with both ends pinned.

2. Calculate the rotation θ of some joint for each of the assumed loads W by use of equations given later.

3. Designate the lowest assumed value of W and the corresponding value of θ as W_1 and θ_1, respectively.

4. Plot the curve of $\frac{\theta - \theta_1}{W - W_1}$ as ordinate against $\theta - \theta_1$ as abscissa. The three assumed loads will give two points on this curve, which are sufficient to establish an approximate value of $W_{crit} - W_1$ and, hence, of W_{crit}. In practical calculations, the actual plotting of the curve can be omitted because the inverse slope $W_{crit} - W_1$ would always be calculated from the numerical values used to plot the curve. If more than three values of W are assumed, however, it may be of interest actually to plot the curve.

ROTATION OF A JOINT

The rotation θ of a joint is easily calculated by the methods of moment distribution. Either of two equations may be used, according to whether the stiffness or the series criterion for stability forms the basis. (See reference 1.)

Stiffness Criterion for Stability

Assume that an external moment M is applied at joint b in figure 1. The moment $-M$ added to balance this joint is distributed among the members as follows:

$$-\frac{M\,S'_{bc_1}}{\Sigma\,S'_{bc}} \quad \text{to member } bc_1$$

$$-\frac{M\,S'_{bc_2}}{\Sigma\,S'_{bc}}, \quad \text{to member } bc_2$$

etc.

The moment distribution analysis is now complete as far as moments at joint b are concerned. (See corresponding discussion in reference 1.)

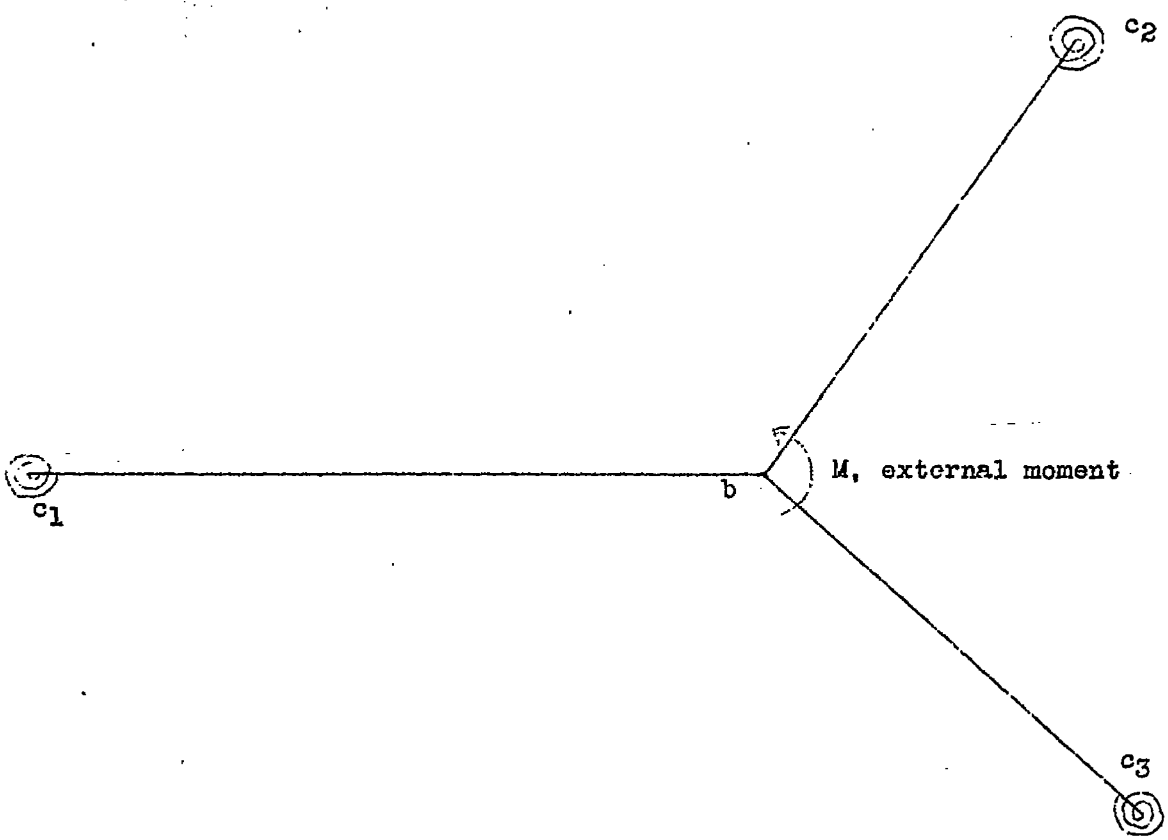


Figure 1

According to the definition of stiffness, the moment distributed to any member must be the rotation of the joint multiplied by the stiffness of the member. Hence,

$- \dfrac{M}{\Sigma S'_{bc}}$ is the rotation in quarter-radians of joint b caused by the external moment M. For the purpose of estimating critical loads, M can have any finite value. For the most convenient value, M = - 1, the rotation θ is, in quarter-radians,

$$\theta = \frac{1}{\Sigma\ S'_{bc}} \tag{5}$$

For stability, the moment in the members and the rotation of the joints must be finite. The stiffness criterion for stability is therefore (reference 1)

$$\Sigma\, S'_{bc} > 0 \tag{6}$$

The condition of neutral stability gives the critical buckling load for the structure and is obtained by setting the stiffness stability factor $\Sigma S'_{bc}$ equal to zero, or

$$\Sigma\, S'_{bc} = 0 \tag{7}$$

Formulas (6) and (7) are also derived in reference 1. The critical load for the structure is obtained by testing equation (7) for different assumed critical loads; the lowest assumed critical load that just satisfies equation (7) is the critical load desired. If the applied load is less than this lowest critical load, the structure is stable; if not, the structure is unstable. The method of estimating the critical load is therefore a tool to aid in finding the lowest critical load that will satisfy equation (7).

Series Criterion for Stability

Assume that an external moment M is applied at joint b in figure 2. From the corresponding analysis in reference 1, it follows that the total moment in members ba at joint b is

$$-\frac{M\,\Sigma\, S'_{ba}}{S_{bc} + \Sigma\, S'_{ba}}\,(1 + r + r^2 + r^3 + \ldots .)$$

or

$$-\frac{M\,\Sigma\, S'_{ba}}{S_{bc} + \Sigma\, S'_{ba}}\,\frac{1}{1 - r}$$

where

$$r = \frac{S_{bc}\,C_{bc}}{S_{bc} + \Sigma\, S'_{ba}}\;\frac{S_{cb}\,C_{cb}}{S_{cd} + \Sigma\, S'_{cd}} \tag{8}$$

cb ?

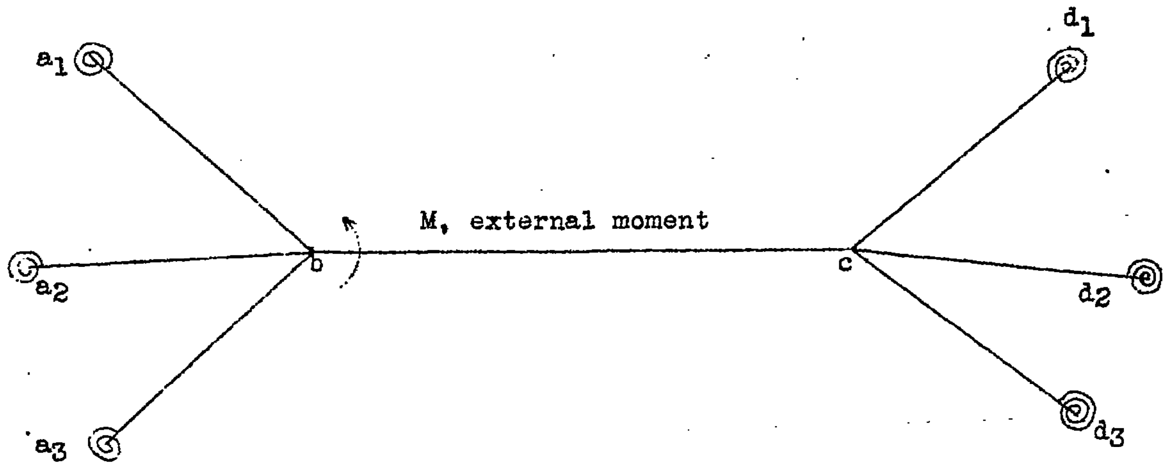


Figure 2

According to the definition of stiffness, the total moment in members ba at joint b must be the rotation of joint b multiplied by the total stiffness of members ba. Hence,

$$-\frac{M}{S_{bc} + \Sigma S'_{ba}} \frac{1}{1 - r}$$

is the rotation in quarter-radians of joint b caused by the external moment M. For the purpose of estimating critical loads, M can have any finite value. For the most convenient value, $M = -1$, the rotation θ is, in quarter-radians,

$$\theta = \frac{1}{S_{bc} + \Sigma S'_{ba}} \frac{1}{1 - r} \tag{9}$$

For stability, the moment in the members and the rotation of the joints must be finite. As stated in reference 1, the series criterion for stability is therefore

$$r < 1 \tag{10}$$

The condition of neutral stability gives the critical buckling load for the structure and is obtained by setting the series stability factor r equal to unity, or

$$r = 1 \tag{11}$$

Formulas (10) and (11) are also derived in reference 1. These expressions are sometimes more convenient to use than the corresponding formulas (6) and (7). In cases when the structure is symmetrical about a joint, the expressions concerned with the stiffness criterion usually involve fewer calculations. When the structure is symmetrical about a member, the formulas concerned with the series criterion offer certain advantages. Experience in the solution of practical problems will dictate which expressions result in fewer calculations. In any case, either set is correct and the method of estimating the critical load is a tool to aid in finding the lowest critical load that will satisfy the equation for neutral stability, either equation (7) or equation (11).

CARRY-OVER FACTOR AND STIFFNESS

In order to check the stability of a group of structural members by use of the previously given formulas, additional equations for the carry-over factor and the stiffness are required.

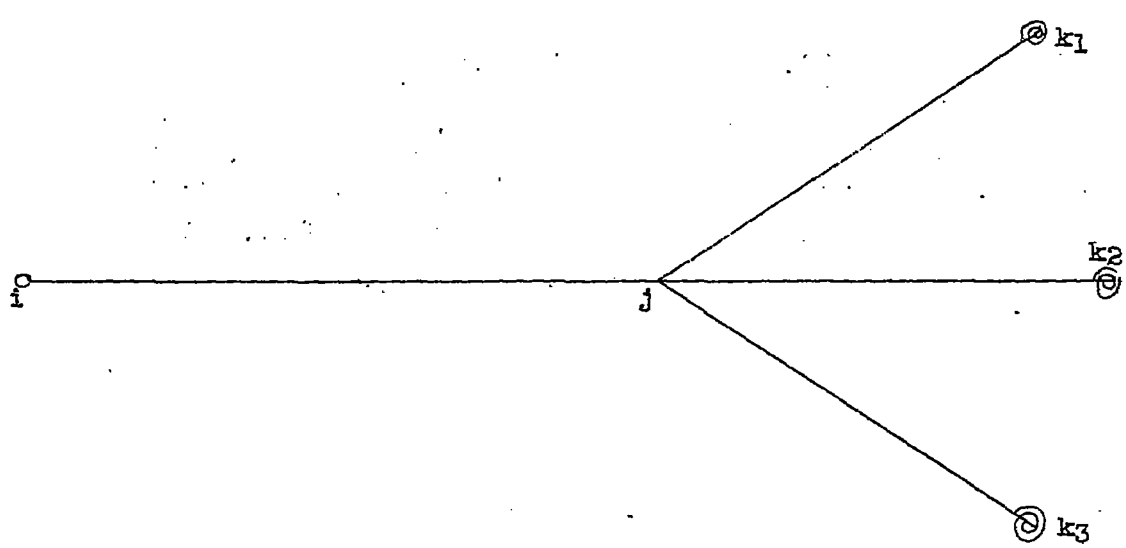


Figure 3

Consider the member ij shown in figure 3, simply supported at i and elastically restrained at j by members jk. The members jk are also elastically restrained at their far ends k. By a moment-distribution analysis given in reference 1, the carry-over factor C'_{ij} is

$$C'_{ij} = C_{ij} \frac{\Sigma S'_{jk}}{S''_{ji} + \Sigma S'_{jk}} \tag{12}$$

and the stiffness S'_{ij} is

$$S'_{ij} = \frac{S''_{ij}}{1 - C_{ji}\, C'_{ij}} \tag{13}$$

Substitution of equation (12) in (13) gives

$$S'_{ij} = \frac{S''_{ij}}{1 - C_{ji}\, C_{ij} \dfrac{\Sigma S'_{jk}}{S''_{ji} + \Sigma S'_{jk}}} \tag{14}$$

For member ij, the limiting values of the carry-over factor and the stiffness given by equations (12) and (14), respectively, are obtained as follows. When the far end j is pinned, there is no elastic restraint at j and $\Sigma S'_{jk} = 0$. For this limiting condition, the carry-over factor $C'_{ij} = C''_{ij} = 0$ and the stiffness $S'_{ij} = S''_{ij}$. When the far end j is fixed, there is complete restraint at j and $\Sigma S'_{jk} = \infty$. For this limiting condition, the carry-over factor $C'_{ij} = C_{ij}$ and the stiffness $S'_{ij} = S_{ij}$, where

$$S_{ij} = \frac{S''_{ij}}{1 - C_{ji}\, C_{ij}} \tag{15}$$

Up to this point, all the equations in this report on stability are general. In nearly all of the cases encountered in practice, however, the cross section and the axial load do not vary along the length of each member. For this special case, $C_{ij} = C_{ji}$, $S''_{ij} = S''_{ji}$, and $S_{ij} = S_{ji}$. In practical problems, the numerical values for these quantities are obtained by use of the tables given in reference 2.

PROBLEMS

The purpose of including problems is to demonstrate the previously described method of estimating critical loads. Six simple problems have been selected to reveal certain characteristics of the method that should be known by the practical engineer using it. In order to show the accuracy of the estimated critical load, the correct value of the critical load for each problem is first established.

The tables of reference 2 were used in the numerical evaluation of the stiffness and the carry-over factor. Although interpolation in these tables is unnecessary for the solution of practical problems, interpolation was used for the solution of these problems to show clearly how the estimated critical load becomes more accurate as the assumed loads W and W_1 approach W_{crit}.

In problems 1 to 4, it is assumed that the members are subjected to low stresses corresponding to the elastic range where the effective modulus $\overline{E}$ is equal to Young's modulus E. In problems 5 and 6, the compression members are loaded above the elastic range where $\overline{E} < E$. In other words, for problems 1 to 4, the compression members lie in the long-column range; whereas, in problems 5 and 6, the compression members lie in the short-column range.

Problem 1

Problem: To calculate the critical load for the pin-end strut shown in figure 4.

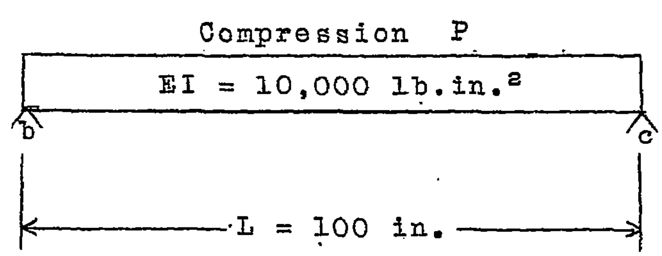


Figure 4. - Problem 1.

The equations concerned with the stiffness criterion for stability are used in the solution of this problem. Imagine the external moment M to be applied at joint b. The correct value of the critical load is therefore the lowest assumed load that will satisfy equation (7). There being only one member bc, the summation sign is omitted. Because this member bc is pinned at the far end c, the single prime on S_{bc} is replaced by a double prime. Thus for this problem, equation (7) becomes

$$S''_{bc} = 0 \tag{16}$$

For member bc

$$EI = 10{,}000 \text{ lb. in.}^2$$

$$L = 100 \text{ in.}$$

Consequently,

$$\frac{L}{j} = \frac{L}{\sqrt{\frac{EI}{P}}} = \sqrt{P}$$

From the tables of reference 2, it is found that the smallest value of P to satisfy equation (16) is the value of P giving $L/j = \pi$. Therefore the correct critical load is

$$P_{crit} = \pi^2 = 9.87 \text{ lb.}$$

which agrees with the value given by the well-known Euler column formula

$$P_{crit} = \frac{\pi^2 EI}{L^2} \tag{17}$$

The estimated value of the critical load is given by the inverse slope of the approximately straight line obtained by plotting $\frac{\theta - \theta_1}{W - W_1}$ as ordinate against $\theta - \theta_1$ as abscissa. For this problem, $W = P$ and equation (5) becomes

$$\theta = \frac{1}{S''_{bc}} \tag{18}$$

The values of θ are given in table I for a series of assumed loads P; and values of $\theta - \theta_1$ and $\frac{\theta - \theta_1}{P - P_1}$ are given for $P_1 = 0$, 3, and 7 pounds. Table I was made extensive in order to show how the estimated critical load is affected by P_1 as well as by the values of P at which the inverse slope is computed.

The approximately straight lines that correspond to $P_1 = 0$, 3, and 7 pounds are plotted in figure 5. Inspection shows the lines corresponding to $P_1 = 0$ and $P_1 = 3$ pounds to be essentially straight. As only two points establish the line for $P_1 = 7$ pounds, no conclusion regarding its straightness is justified.

If $P_1 = 0$, then the inverse slope between $P = 1$ and $P = 2$ pounds is (see table I)

$$P_{crit} - P_1 = \frac{0.002202 - 0.000983}{0.001101 - 0.000983} = 10.33 \text{ lb.}$$

from which

$$P_{crit} = (P_{crit} - P_1) + P_1 = 10.33 + 0 = 10.33 \text{ lb.}$$

The results of a number of calculations of this type for other values of P and P_1 are given in table II. Inspection of this table shows that, for any value of P_1, the estimated critical load becomes more accurate as the values of P between which the inverse slope is calculated approach P_{crit}. The accuracy is also increased as P_1 approaches P_{crit}.

Problems 2, 3, and 4

The purpose of problems 2, 3, and 4 is to study the effect of the tension in tension members on the estimated critical load for a structure. In these problems, end b of the strut used in problem 1 is restrained against rotation by the adjacent member ba, which is the same size as member bc. (See fig. 6.) In problem 2, member ba has zero axial load. In problems 3 and 4, there is axial tension in ba of magnitude P and $3P$, respectively. In each problem, the compression in member bc is of magnitude P.

Problem	Member ba	Member bc
2	Zero axial load	Compression P
3	Tension P	Compression P
4	Tension 3P	Compression P

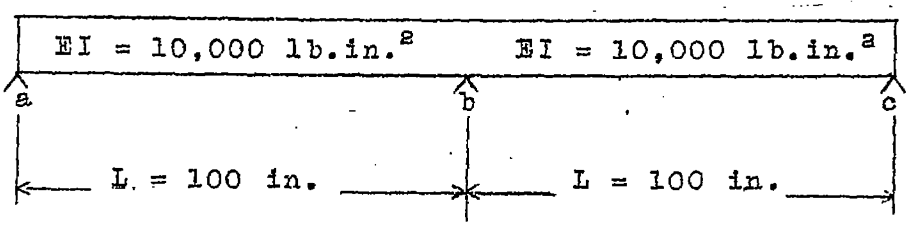


Figure 6. - Problems 2, 3, and 4.

Imagine the external moment M to be applied at joint b. The correct value of the critical load is the lowest assumed load that will satisfy equation (7) which becomes, for problems 2, 3, and 4,

$$S''_{ba} + S''_{bc} = 0 \qquad (19)$$

For each of members ba and bc in problems 2, 3, and 4,

$$EI = 10{,}000 \text{ lb. in.}^2$$

$$L = 100 \text{ in.}$$

On calculation of the values of L/j for each span in each of the problems, it is found by trial that the lowest value of P satisfying equation (19), or P_{crit}, is

Problem	P_{crit} (lb.)
2	13.89
3	15.41
4	16.93

The estimated value of the critical load is given by the inverse slope of the approximately straight line obtained by plotting $\frac{\theta - \theta_1}{P - P_1}$ as ordinate against $\theta - \theta_1$ as abscissa. For problems 2, 3, and 4, equation (5) becomes

$$\theta = \frac{1}{S''_{ba} + S''_{bc}} \qquad (20)$$

In tables III, IV, and V, the assumed values of P and the corresponding values of θ are given for problems 2, 3, and 4, respectively. The curves established by the data in these tables are plotted in figures 7, 8, and 9, respectively. A summary of the corresponding estimated critical loads for each problem is given in tables VI, VII, and VIII.

In figure 7, the curve for $P_1 = 0$ is noticeably concave upward; whereas, in figures 8 and 9, this curve is definitely concave downward. This change from concave upward to concave downward is caused by the tension in member ba, which results in an overestimation of the critical load when the tension in member ba is zero but an underestimation when the tension is equal to P and $3P$. (See tables VI, VII, and VIII.) These same conclusions hold in a lesser degree when $0 < P_1 < P_{crit}$.

When the method of estimating critical loads is applied in the solution of practical problems, it is desirable to know whether the true critical load is overestimated or underestimated. From problems 1 and 2 it is concluded that, in the absence of tension members, the estimated critical loads are all greater than the true critical load. (See tables II and VI.) From problems 3 and 4 it is concluded that, in the presence of tension members, the estimated critical loads are all less than the true critical load. (See tables VII and VIII.) The region within which all estimated critical loads are in good agreement with the true critical load cannot be definitely established in the general case.

Qualitatively, the region of transition from overestimating to underestimating the critical load can be established by noting the trends in problems 2, 3, and 4. In problem 2, the poorest estimate of the critical load (small values of P and P_1) is 12.8 percent on the unsafe side. In problems 3 and 4, the poorest estimates are 69.6 and 122.7 percent, respectively, on the safe side. In problem 2, no tension member is present. In problem 3, the size, the axial load, and the number of the tension members are the same as for the compression members. In problem 4, the axial load in the tension member is three times the axial load in the compression member. It is therefore concluded that the transition from overestimating to underestimating the critical load will occur when the size, the axial load, or the number of the tension members is small relative to the compression members.

When all members in a given problem are compression members or when the effects of tension are neglected in the calculation, as is sometimes done in stability problems of this type, the agreement of the estimated critical load with the calculated critical load will be as good as that found in problems 1 and 2. When the effect of tension in tension members is considered, the precision of the estimated critical load can be determined qualitatively by reference to problems 2, 3, and 4.

As in the case of problem 1, the agreement of the estimated critical load with the calculated critical load for problems 2, 3, and 4 becomes closer as the values of P between which the inverse slope is calculated, approach P_{crit}. The precision also increases as P_1 approaches P_{crit}. (See tables VI, VII, and VIII.)

THE EFFECTIVE MODULUS

Before the theory of this report can be applied to problems involving compression members that are stressed beyond the elastic range, as in problems 5 and 6, it is necessary to introduce an effective modulus $\bar{E}$ so designed that the results will be in good agreement with the accepted column formulas.

Compression Members

Most engineers are familiar with the origin of the accepted column curve for a given material. At low stresses (stresses less than about one-half the yield point of the material), the column strength is given by the Euler formula. At high stresses, laboratory tests always show that the column strength falls short of the value given by the Euler formula. An empirical straight line or a parabolic curve is sometimes used to give the column strength within this range.

The theory of this report gives a buckling load that is analogous to the strength given by the Euler column formula. As in the case of the Euler formula, these loads would not check experimental values at high stresses. A reduced strength must therefore be calculated consistent with the accepted column formula for the material of which the members are composed. These calculations are best made by use of the effective modulus $\overline{E} = \tau E$.

Consider the case of an ordinary column. If the Euler formula is written

$$\frac{P}{A} = \frac{\pi^2 \tau E}{\left(\frac{L}{\sqrt{c}\,\rho}\right)^2} \tag{21}$$

it will give the strength at both low and high stresses. At low stresses, $\tau = 1$; whereas, at high stresses, $\tau < 1$. The problem is to determine how the effective modulus τE varies with the stress P/A.

If equation (21) is solved for τE, the following equation is obtained

$$\tau E = \frac{1}{\pi^2} \frac{P}{A} \frac{1}{c} \left(\frac{L}{\rho}\right)^2 \tag{22}$$

The accepted column formula for any material is always given in terms of the effective slenderness ratio $(L/\sqrt{c}\,\rho)$. Thus, if any one of these formulas is solved for $(L/\sqrt{c}\,\rho)$ and this value is substituted into equation (22), there results an equation for the effective modulus τE that is a function of the stress P/A.

For example, consider the case of S.A.E. 1025 steel. The column formulas for this material are:

For $\frac{P}{A} < 18{,}000$ lb. per sq. in.

$$\frac{P}{A} = \frac{\pi^2 E}{\frac{1}{c}\left(\frac{L}{\rho}\right)^2} \tag{23}$$

For $36{,}000 > \frac{P}{A} > 18{,}000$ lb. per sq. in.,

$$\frac{P}{A} = 36{,}000 - 1.172\,\frac{1}{c}\left(\frac{L}{\rho}\right)^2 \tag{24}$$

If equations (23) and (24) are solved for $(L/\sqrt{c}\ \rho)$ and these values are substituted into equation (22), the following values are obtained for the effective modulus $\overline{E} = \tau E$.

For $\frac{P}{A} < 18{,}000$ lb. per sq. in.,

$$\overline{E} = \tau E = E \tag{25}$$

For $36{,}000 > \frac{P}{A} > 18{,}000$ lb. per sq. in.,

$$\overline{E} = \tau E = \frac{1}{\pi^2}\,\frac{P}{A}\left(\frac{36{,}000 - \frac{P}{A}}{1.172}\right) \tag{26}$$

Equation (25) shows that, in the long-column or elastic range, $\overline{E} = E$. Equation (26) shows that, in the short-column range, $\overline{E}$ is a function of the stress P/A in the member and is in no way dependent upon the stiffness or end fixity of the member.

When the compression members of the structure are stressed beyond the elastic range, the methods outlined in this report can also be used to calculate the critical load. The procedure is the same as in problems 1 to 4 except that, for each assumed load W on the structure, there is a different value of the effective modulus $\overline{E}$. These values of $\overline{E}$ are obtained by use of equations (25) and (26) if the material is S.A.E. 1025 steel. For any other material, corresponding equations can be derived.

Tension Members

When the effect of axial load in the tension members is considered, the variation of $\overline{E}$ with stress for tension members can be established, theoretically, by the use of the double-modulus theory of bending and of the stress-strain curve of the material. For such calculations, however, the stress-strain curve must be accurately drawn to a suitable scale. In the absence of a known or a calculated variation of $\overline{E}$ with stress, the following approximate method can be used to establish $\overline{E}$ for tension members:

1. When the stress is less than the maximum allowed for a column of the same material, use the same

values of $\bar{E}$ for tension as for compression at the same stress.

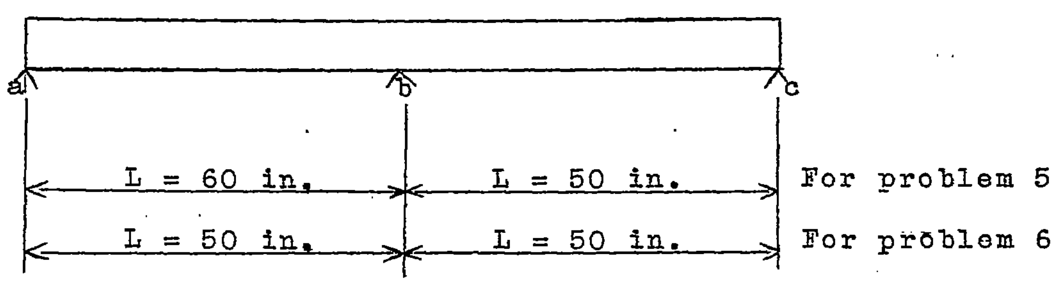
2. When the stress is greater than the maximum allowed for a column of the same material, assume that $\bar{E} = 0$.

The values of $\bar{E}$ for tension members obtained by this method will be conservative. Whether or not they are too conservative is a matter to be settled by tests. In the regions of yield point and of maximum tensile strength, the flatness of the stress-strain curve will certainly cause $\bar{E}$ to approach zero. Because the maximum stress allowed in columns is closely associated with the yield point, this method offers a convenient solution of $\bar{E}$ for tension members.

Problems 5 and 6

The purpose of problems 5 and 6 is to show that the method of estimating critical loads presented in this paper gives good results when the compression members lie within the short-column range. Except for the different dimensions and the fact that the members with axial load are stressed beyond the elastic range, these problems are similar to problems 2 and 3, respectively.

Problem	Member ba	Member bc
5	Zero axial load	Compression P
6	Tension P	Compression P



Material: S.A.E. 1025 steel tube continuous from a to c with the following dimensions:

Diameter, d	1.625 in.
Wall thickness, t	.065 in.
Area, A	.3186 sq. in.
Moment of inertia, I	.0970 in.^{4}

Figure 10. - Problems 5 and 6.

The essential dimensions for problems 5 and 6 are given in figure 10. The effective modulus $\bar{E}$ for any member is a function of the stress P/A in that member. The numerical value of $\bar{E}$ for any assumed load P is therefore given by equations (25) and (26), the material being S.A.E. 1025 steel. By the same methods as used in the solution of problems 2 and 3, it is found that the lowest value of P to satisfy equation (19), or P_{crit}, is

Problem	P_{crit} (lb.)
5	9,420
6	9,510

The necessary calculations for estimating the critical loads for problems 5 and 6 were made by the same methods used for problems 2 and 3 except that in the calculation of the stiffness of the members the effective modulus $\bar{E}$ was used in place of Young's modulus E. The results of these calculations are given in tables IX to XIV.

In figures 11 and 12, $\dfrac{\theta - \theta_1}{P - P_1}$ is plotted against $\theta - \theta_1$ for problems 5 and 6. It is preferable, however, to compare the results given in tables XI and XIV rather than to draw conclusions from figures 11 and 12.

In problem 5, the axial load in member ba is zero; whereas, in problem 6, member ba is subjected to axial tension equal to the axial compression of member bc. Comparison of the estimated critical loads for each of these problems (tables XI and XIV) shows that the critical load is usually, but not always, overestimated when the tension in member ba is zero and is usually, but not always, underestimated when the tension in member ba is equal to the compression in member bc. These same conclusions were found for problems 2 and 3.

Comparison of the precision of the estimated critical loads for problems 5 and 6 (tables XI and XIV, respectively) with the precision of the estimated critical loads for problems 2 and 3 (tables VI and VII, respectively) is not justified. For problems 2 and 3, the series of estimated critical loads are based upon values of P taken at intervals of roughly 10 percent of P_{crit}; whereas, for problems 5 and 6, this interval was not maintained. When the members lie in the short-column range, an estimated critical load based upon small values of P, which lie in the elastic range, gives an estimated critical load much higher than the true critical load. Consequently, in problems 5 and 6, the assumed values of P for which the estimated critical load was obtained were made to correspond to values of P/A that are associated with the short-column range.

In the solution of any problem, it is necessary only that the assumed loads be less than the true critical load. As the assumed loads approach the true critical load, the precision of the estimated critical load is increased. (See tables XI and XIV.) It is therefore desirable to exercise the best judgment possible in the selection of the assumed loads. In any case, however, the method of estimating the critical load as described in this paper should be regarded as a tool to be used in finding the lowest critical load that will satisfy the equation for neutral stability. If it is desired that the estimated critical load be conservative rather than err on the unsafe side, the effect of the axial load in the tension members should be considered in the calculation.

CONCLUSIONS

1. If the distribution of the total load W on the structure does not change as W increases, then the axial load in each member is proportional to W. Thus, if $\frac{\theta - \theta_1}{W - W_1}$ is plotted as ordinate against $\theta - \theta_1$ as abscissa, the curve obtained when W approaches W_{crit} is essentially a straight line the inverse slope of which is $W_{crit} - W_1$, where

θ is the rotation of a joint under the moment M at load W on the structure.

θ_1 and W_1, initial values of θ and W, respectively.

W_{crit}, lowest critical load.

and

$$W_1 < W < W_{crit}$$

Thus, if simultaneous values of load and rotation are plotted as just described beginning with W_1 as the initial load, the value of $W_{crit} - W_1$ is easily obtained. The value of W_{crit} is then given by the equation

$$W_{crit} = (W_{crit} - W_1) + W_1$$

2. The rotation θ of a joint can be calculated by the methods of moment distribution. The equation to be used depends on whether the stiffness or the series criterion for stability forms the basis of the calculation

3. For loads within the elastic range, the estimated critical load more closely agrees with the calculated critical load as the values of W between which the inverse slope is calculated approach W_{crit}. The agreement is also closer as W_1 approaches W_{crit}.

4. For loads beyond the elastic range, the results of computation have shown that conclusion 3 usually, but not always, applies. In cases where it does not apply, the errors are of the order of a fraction of 1 percent. For practical design calculations, conclusion 3 therefore holds for loads beyond the elastic range as well as for loads within the elastic range.

5. When all members in a given problem are compression members or when the effects of tension are neglected, as is sometimes done in practical calculations, the calculated critical load is overestimated. When the effect of tension in the tension members is considered, however, the calculated critical load is underestimated. The region within which all estimated critical loads are in good agreement with the calculated critical load cannot be definitely established in the general case. The transition from overestimating to underestimating the calculated critical load tends to occur, however, when the size, the axial load, or the number of tension members is small relative to the compression members. In many practical problems, the precision with which the estimated critical load agrees with the calculated critical load can be qualitatively determined by reference to the problems of this report.

6. The method of estimating the critical load should always be regarded as a tool to aid in finding the lowest

load that satisfies the equation for neutral stability. This lowest load is the calculated critical load for the problem.

Langley Memorial Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., June 8, 1939.

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TABLE I

Calculated Data for Problem 1

P	θ	P_1 = 0, θ_1 = 0.013333		P_1 = 3 lb., θ_1 = 0.017090		P_1 = 7 lb., θ_1 = 0.033668	
(lb.)	(radian/4)	$\theta-\theta_1$ (radian/4)	$\frac{\theta-\theta_1}{P-P_1}$ (radian/4 lb.)	$\theta-\theta_1$ (radian/4)	$\frac{\theta-\theta_1}{P-P_1}$ (radian/4 lb.)	$\theta-\theta_1$ (radian/4)	$\frac{\theta-\theta_1}{P-P_1}$ (radian/4 lb.)
0	0.013333	0	0				
1	.014316	.000983	.000983				
2	.015535	.002202	.0011010				
3	.017090	.003757	.0012523	0			
4	.019153	.005820	.0014550	.002063	0.002063		
5	.022036	.008703	.0017406	.004946	.002473		
6	.026369	.013036	.0021727	.009279	.003093		
7	.033668	.020335	.0029050	.016578	.0041445	0	
8	.048669	.035336	.0044170	.031579	.0063158	.015001	0.15001
9	.097982	.084649	.0094054	.080892	.013482	.064314	.032157

TABLE IV

Calculated Data for Problem 3

P	θ	P_1 = 0, θ_1 = 0.0066667		P_1 = 4.8 lb., θ_1 = 0.0069891		P_1 = 11.2 lb., θ_1 = 0.0099641	
(lb.)	(radian/4)	$\theta-\theta_1$ (radian/4)	$\frac{\theta-\theta_1}{P-P_1}$ (radian/4 lb.)	$\theta-\theta_1$ (radian/4)	$\frac{\theta-\theta_1}{P-P_1}$ (radian/4 lb.)	$\theta-\theta_1$ (radian/4)	$\frac{\theta-\theta_1}{P-P_1}$ (radian/4 lb.)
0	0.0066667	0					
1.6	.0066993	.0000326	0.000020375				
3.2	.0068023	.0001356	.000042375				
4.8	.0069891	.0003224	.000067167	0			
6.4	.0072902	.0006235	.000097422	.0003011	0.00018819		
8.0	.0077664	.0010997	.00013746	.0007773	.00024291		
9.6	.0085470	.0018803	.00019586	.0015579	.00032456		
11.2	.0099641	.0032974	.00029441	.0029750	.00046484	0	
12.8	.013166	.0064997	.00050777	.0061773	.00077216	.0032023	0.0020014
14.4	.026515	.019848	.0013783	.019526	.0020339	.016551	.0051720

TABLE III

Calculated Data for Problem 2

P	θ	P_1 = 0, θ_1 = 0.0066667		P_1 = 4.2 lb., θ_1 = 0.0079434		P_1 = 9.8 lb., θ_1 = 0.013181	
(lb.)	(radian/4)	$\theta-\theta_1$ (radian/4)	$\frac{\theta-\theta_1}{P-P_1}$ (radian/4 lb.)	$\theta-\theta_1$ (radian/4)	$\frac{\theta-\theta_1}{P-P_1}$ (radian/4 lb.)	$\theta-\theta_1$ (radian/4)	$\frac{\theta-\theta_1}{P-P_1}$ (radian/4 lb.)
0	0.0066667	0					
1.4	.0070077	.0003410	0.00024357				
2.8	.0074229	.0007562	.00027007				
4.2	.0079434	.0012767	.00030398	0			
5.6	.0086213	.0019546	.00034904	.0006779	0.00048421		
7.0	.0095509	.0028842	.00041203	.0016075	.00057411		
8.4	.010921	.0042544	.00050648	.0029777	.00070898		
9.8	.013181	.0065146	.00066476	.0052379	.00093534	0	
11.2	.017715	.011048	.00098646	.0097717	.0013960	.0045338	0.0032384
12.6	.031927	.025260	.0020048	.023984	.0028552	.018746	.0066950

TABLE V

Calculated Data for Problem 4

P	θ	P_1 = 0, θ_1 = 0.0066667		P_1 = 5.1 lb., θ_1 = 0.0056844		P_1 = 11.9 lb., θ_1 = 0.0067033	
(lb.)	(radian/4)	$\theta-\theta_1$ (radian/4)	$\frac{\theta-\theta_1}{P-P_1}$ (radian/4 lb.)	$\theta-\theta_1$ (radian/4)	$\frac{\theta-\theta_1}{P-P_1}$ (radian/4 lb.)	$\theta-\theta_1$ (radian/4)	$\frac{\theta-\theta_1}{P-P_1}$ (radian/4 lb.)
0	0.0066667	0					
1.7	.0061155	-.0005512	-0.00032424				
3.4	.0058231	-.0008436	-.00024812				
5.1	.0056844	-.0009823	-.00019261	0			
6.8	.0056686	-.0009981	-.00014678	-.0000158	-0.0000092[illegible]		
8.5	.0057820	-.0008847	-.00010408	.0000976	.000028706		
10.2	.0060761	-.0005906	-.00005790	.0003917	.000076804		
11.9	.0067033	.0000366	.0000030[illegible]	.0010189	.00014984	0	
13.6	.0079834	.0013167	.00009682	.0022990	.00027047	.0012801	0.0007530
15.3	.013765	.0070479	.00046392	.0080802	.00079218	.0070613	.0020769

TABLE II

Summary of Estimated Critical Loads

for Problem 1

[P_{crit} (theoretical) = 9.87 lb.]

P_1 (lb.)	Values of P between which slope is calculated (lb.)	P_{crit} (estimated) (lb.)	$\frac{P_{crit} \text{ (estimated)}}{P_{crit} \text{ (theoretical)}}$
0	1 and 2	10.33	1.047
	4 and 5	10.10	1.023
	8 and 9	9.89	1.002
3	4 and 5	10.03	1.016
	8 and 9	9.88	1.001
7	8 and 9	9.87	1.000

TABLE VI

Summary of Estimated Critical Loads for Problem 2

[P_{crit} (theoretical) = 13.89 lb.]

P_1 (lb.)	Values of P between which slope is calculated (lb.)	P_{crit}(estimated) (lb.)	$\frac{P_{crit}\text{(estimated)}}{P_{crit}\text{(theoretical)}}$
0	1.4 and 2.8	15.67	1.128
	5.6 and 7.0	14.76	1.063
	11.2 and 12.6	13.96	1.005
4.2	5.6 and 7.0	14.54	1.047
	11.2 and 12.6	13.94	1.004
9.8	11.2 and 12.6	13.91	1.001

TABLE VII

Summary of Estimated Critical Loads for Problem 3

[P_{crit} (theoretical) = 15.41 lb.]

P_1 (lb.)	Values of P between which slope is calculated (lb.)	P_{crit}(estimated) (lb.)	$\frac{P_{crit}\text{(estimated)}}{P_{crit}\text{(theoretical)}}$
0	1.6 and 3.2	4.68	0.304
	6.4 and 8.0	11.89	.772
	12.8 and 14.4	15.33	.995
4.8	6.4 and 8.0	13.50	0.876
	12.8 and 14.4	15.38	.998
11.2	12.8 and 14.4	15.41	1.000

TABLE VIII

Summary of Estimated Critical Loads for Problem 4

[P_{crit} (theoretical) = 16.93 lb.]

P_1 (lb.)	Values of P between which slope is calculated (lb.)	P_{crit}(estimated) (lb.)	$\frac{P_{crit}\text{(estimated)}}{P_{crit}\text{(theoretical)}}$
0	1.7 and 3.4	-3.84	-0.227
	6.8 and 8.5	2.66	.157
	13.6 and 15.3	15.75	.930
5.1	6.8 and 8.5	8.08	0.477
	13.6 and 15.3	16.18	.956
11.9	13.6 and 15.3	16.27	0.961

TABLE XI

Summary of Estimated Critical Loads for Problem 5

[P_{crit} (theoretical) = 9,420 lb.]

P_1 (lb.)	Values of P between which slope is calculated (lb.)	P_{crit}(estimated) (lb.)	$\frac{P_{crit}\text{(estimated)}}{P_{crit}\text{(theoretical)}}$
0	5,734.8 and 6,372.0	11,995	1.273
	7,009.2 and 7,646.4	9,792.0	1.039
	8,283.6 and 8,920.8	9,602.1	1.019
5,734.8	7,009.2 and 7,646.4	9,263.2	.983
	8,283.6 and 8,420.8	9,556.5	1.014
7,646.4	8,283.6 and 8,920.8	9,565.8	1.015

TABLE IX

Stiffness Values for Problem 5

Member bc						
P	$\frac{P}{A}$	E	$\frac{EI}{L}$	$\left(\frac{L}{J}\right)_{eff}$	S'_{bc}	$S''_{bc}+S''_{ba}$
0	0	28,000,000	54,359	0	40,769	74,739
5734.8	18,000	28,012,000	54,382	2.2963	23,546	57,512
6372.0	20,000	27,664,000	53,707	2.4556	20,563	54,533
7009.2	22,000	26,627,000	51,693	2.6038	16,416	50,386
7646.4	24,000	24,898,000	48,336	2.8124	10,415	44,385
8283.6	26,000	22,478,000	43,638	3.0808	2,031.5	36,002
8920.8	28,000	19,365,000	37,596	3.4444	10,922	23,048

$$E = \frac{P}{A}\left[\frac{36,000-\frac{P}{A}}{1.172\,\pi^2}\right]$$

For member ba: $P=0$, $\frac{P}{A}=0$, $E=28\times10^6$ lb. per sq in., $\left(\frac{L}{J}\right)_{eff}=0$, and $S''_{ba}=3.397\times10^4$ lb. in.

TABLE XII

Stiffness Values for Problem 6

Members ba and bc					S''_{ba}	S''_{bc}	$S''_{ba}+S''_{bc}$
P	$\frac{P}{A}$	E	$\frac{EI}{L}$	$\left(\frac{L}{J}\right)_{eff}$			
0	0	28,000,000	54,359	0	40,769	40,769	81,538
5734.8	18,000	28,012,000	54,382	2.2963	53,404	23,542	76,946
6372.0	20,000	27,664,000	53,707	2.4556	53,712	20,563	74,275
7009.2	22,000	26,627,000	51,693	2.6038	53,631	16,416	70,047
7646.4	24,000	24,898,000	48,336	2.8124	52,034	10,415	62,449
8283.6	26,000	22,478,000	43,638	3.0808	49,625	2,031.5	51,657
8920.8	28,000	19,365,000	37,596	3.4444	45,130	-10,922	34,208

$$E = \frac{P}{A}\left[\frac{36,000-\frac{P}{A}}{1.172\,\pi^2}\right]$$

TABLE X

Calculated Data for Problem 5

P	θ	$P_1=0$, $\theta_1=0.13380\times10^{-4}$		$P_1=5734.8$ lb, $\theta_1=0.17388\times10^{-4}$		$P_1=7646.4$ lb. $\theta_1=0.22530\times10^{-4}$	
		$\theta-\theta_1$	$\frac{\theta-\theta_1}{P-P_1}$	$\theta-\theta_1$	$\frac{\theta-\theta_1}{P-P_1}$	$\theta-\theta_1$	$\frac{\theta-\theta_1}{P-P_1}$
(lb.)	$\left(\frac{\text{radian}}{4}\right)$	$\left(\frac{\text{radian}}{4}\right)$	$\left(\frac{\text{radian}}{4\text{ lb.}}\right)$	$\left(\frac{\text{radian}}{4}\right)$	$\left(\frac{\text{radian}}{4\text{ lb.}}\right)$	$\left(\frac{\text{radian}}{4}\right)$	$\left(\frac{\text{radian}}{4\text{ lb.}}\right)$
0	0.13380×10^{-4}	0					
5734.8	.17388	$.04008\times10^{-4}$	0.06989×10^{-8}	0			
6372.0	.18338	.04958	.07781	$.00950\times10^{-4}$	0.14909×10^{-8}		
7009.2	.19847	.06467	.09226	.02459	.19295		
7646.4	.22530	.09150	.11966	.05142	.26899	0	
8283.6	.27776	.14396	.17379	.10388	.40756	$.05246\times10^{-4}$	0.82329×10^{-8}
8920.8	.43388	.30008	.33638	.26000	.81607	.20858	1.6367

TABLE XIII

Calculated Data for Problem 6

P	θ	$P_1=0$, $\theta_1=0.12264\times10^{-4}$		$P_1=5734.8$ lb, $\theta_1=0.12996\times10^{-4}$		$P_1=7646.4$ lb, $\theta_1=0.16013\times10^{-4}$	
		$\theta-\theta_1$	$\frac{\theta-\theta_1}{P-P_1}$	$\theta-\theta_1$	$\frac{\theta-\theta_1}{P-P_1}$	$\theta-\theta_1$	$\frac{\theta-\theta_1}{P-P_1}$
(lb.)	$\left(\frac{\text{radian}}{4}\right)$	$\left(\frac{\text{radian}}{4}\right)$	$\left(\frac{\text{radian}}{4\text{ lb.}}\right)$	$\left(\frac{\text{radian}}{4}\right)$	$\left(\frac{\text{radian}}{4\text{ lb.}}\right)$	$\left(\frac{\text{radian}}{4}\right)$	$\left(\frac{\text{radian}}{4\text{ lb.}}\right)$
0	0.12264×10^{-4}	0					
5734.8	.12996	$.00732\times10^{-4}$	0.012764×10^{-8}	0			
6372.0	.13463	.01199	.018817	$.00467\times10^{-4}$	0.073289×10^{-8}		
7009.2	.14276	.02012	.028705	.01280	.10044		
7646.4	.16013	.03749	.049030	.03017	.15783	0	
8283.6	.19358	.07094	.085639	.06362	.24961	$.03345\times10^{-4}$	0.52496×10^{-8}
8920.8	.29233	.16969	.190218	.16237	.50964	.15220	1.03735

TABLE XIV

Summary of Estimated Critical Loads

for Problem 6

[P_{crit} (theoretical) = 9,510 lb.]

P_1 (lb.)	Values of P between which slope is calculated (lb.)	P_{crit} (estimated) (lb.)	$\frac{P_{crit} \text{ (estimated)}}{P_{crit} \text{ (theoretical)}}$
0	5,734.8 and 6,372.0	7,715.2	0.811
	7,009.2 and 7,646.4	8,546.1	.899
	8,283.6 and 8,920.8	9,442.6	.993
5,734.8	7,009.2 and 7,646.4	8,761.5	.921
	8,283.6 and 8,920.8	9,532.4	1.002
7,646.4	8,283.6 and 8,920.8	9,573.6	1.007

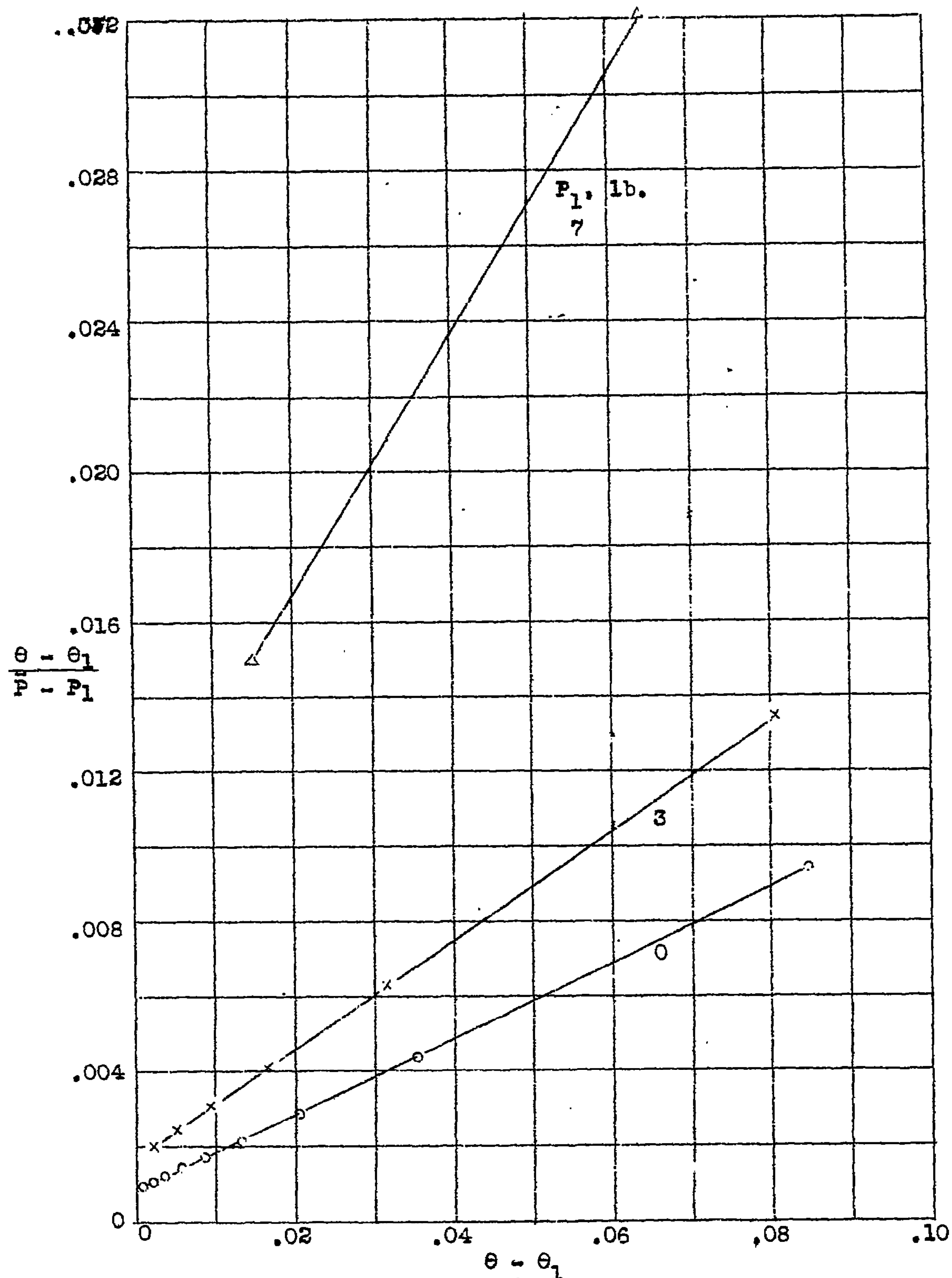

Figure 5.- Variation of $\frac{\theta - \theta_1}{P - P_1}$ with $\theta - \theta_1$ for problem 1.
(See table I.)

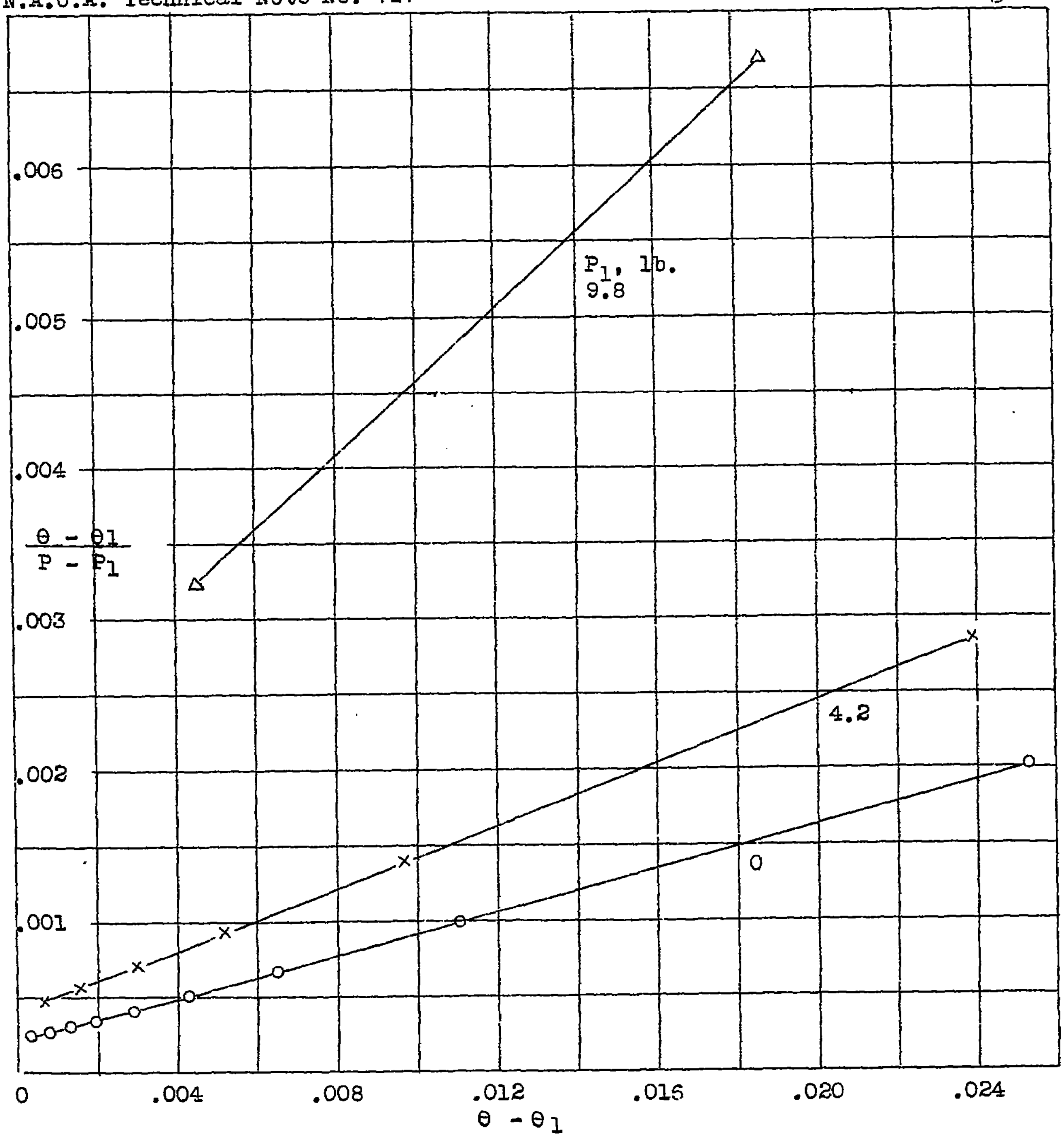

Figure 7.- Variation of $\frac{\theta-\theta_1}{P-P_1}$ with $\theta-\theta_1$ for problem 2. (See table III.)

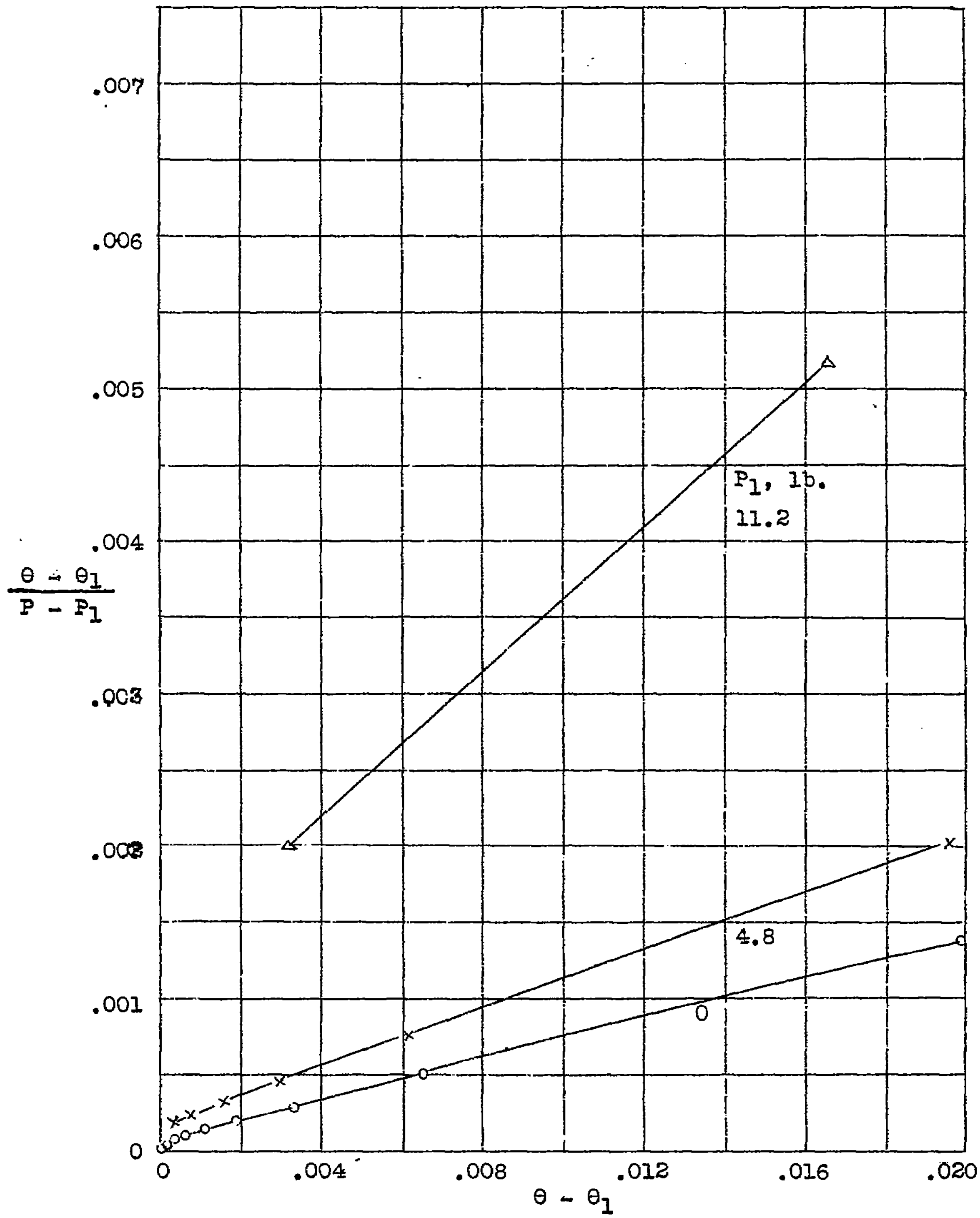

Figure 8.- Variation of $\frac{\theta - \theta_1}{P - P_1}$ with $\theta - \theta_1$ for problem 3. (See table IV.)

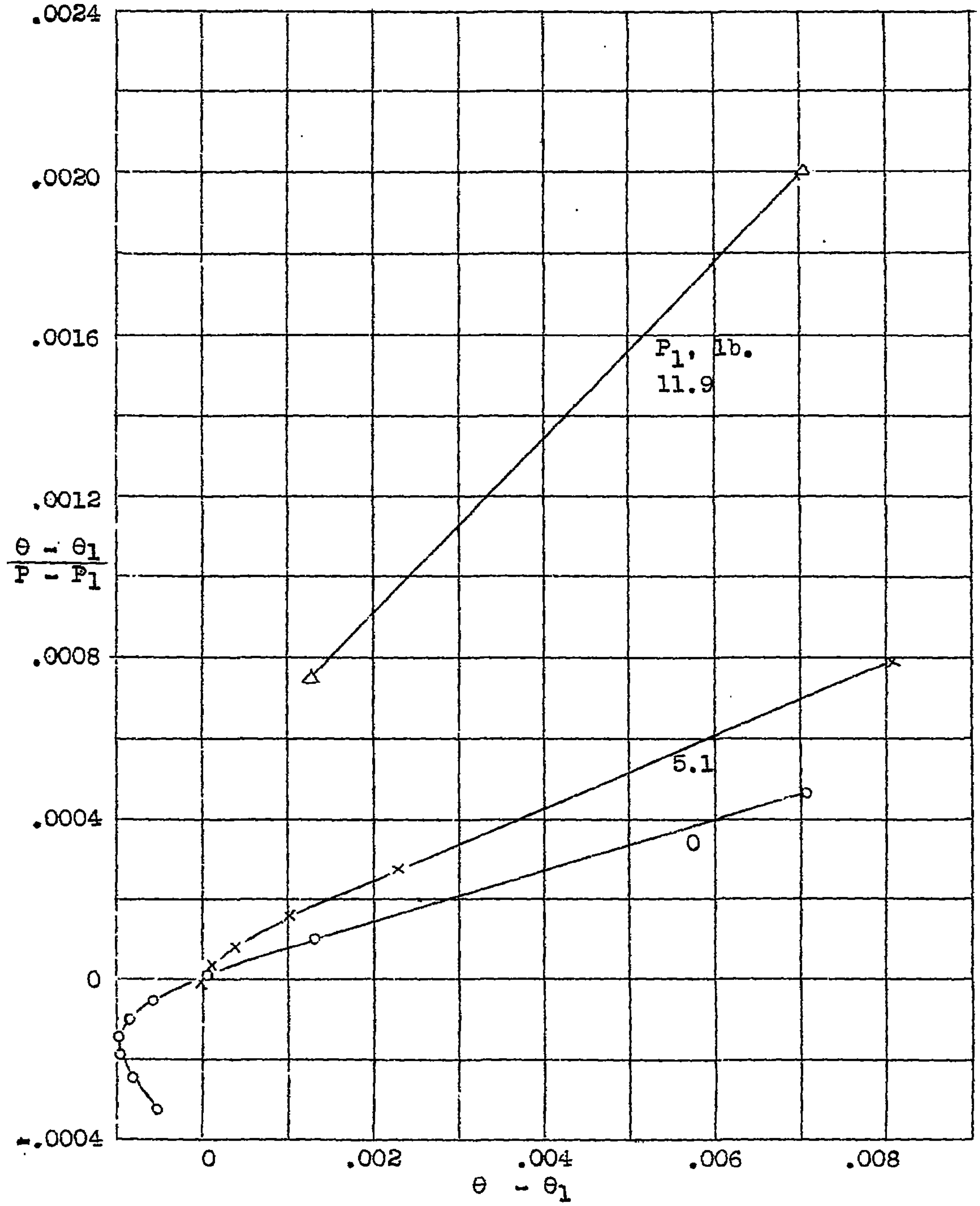

Figure 9.- Variation of $\frac{\theta - \theta_1}{P - P_1}$ with $\theta - \theta_1$ for problem 4. (See table V.)

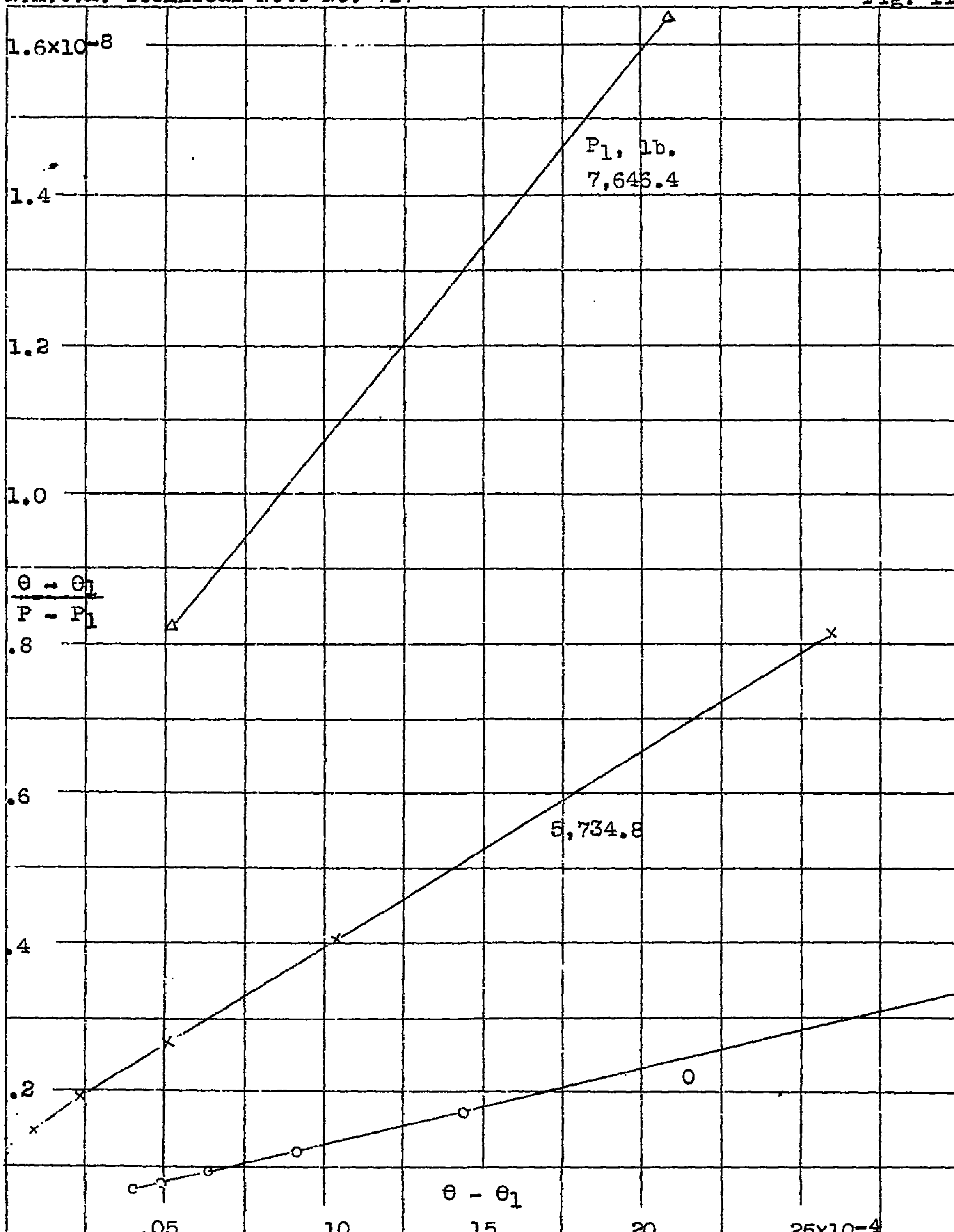

Figure 11.- Variation of $\frac{\theta - \theta_1}{P - P_1}$ with $\theta - \theta_1$ for problem 5.

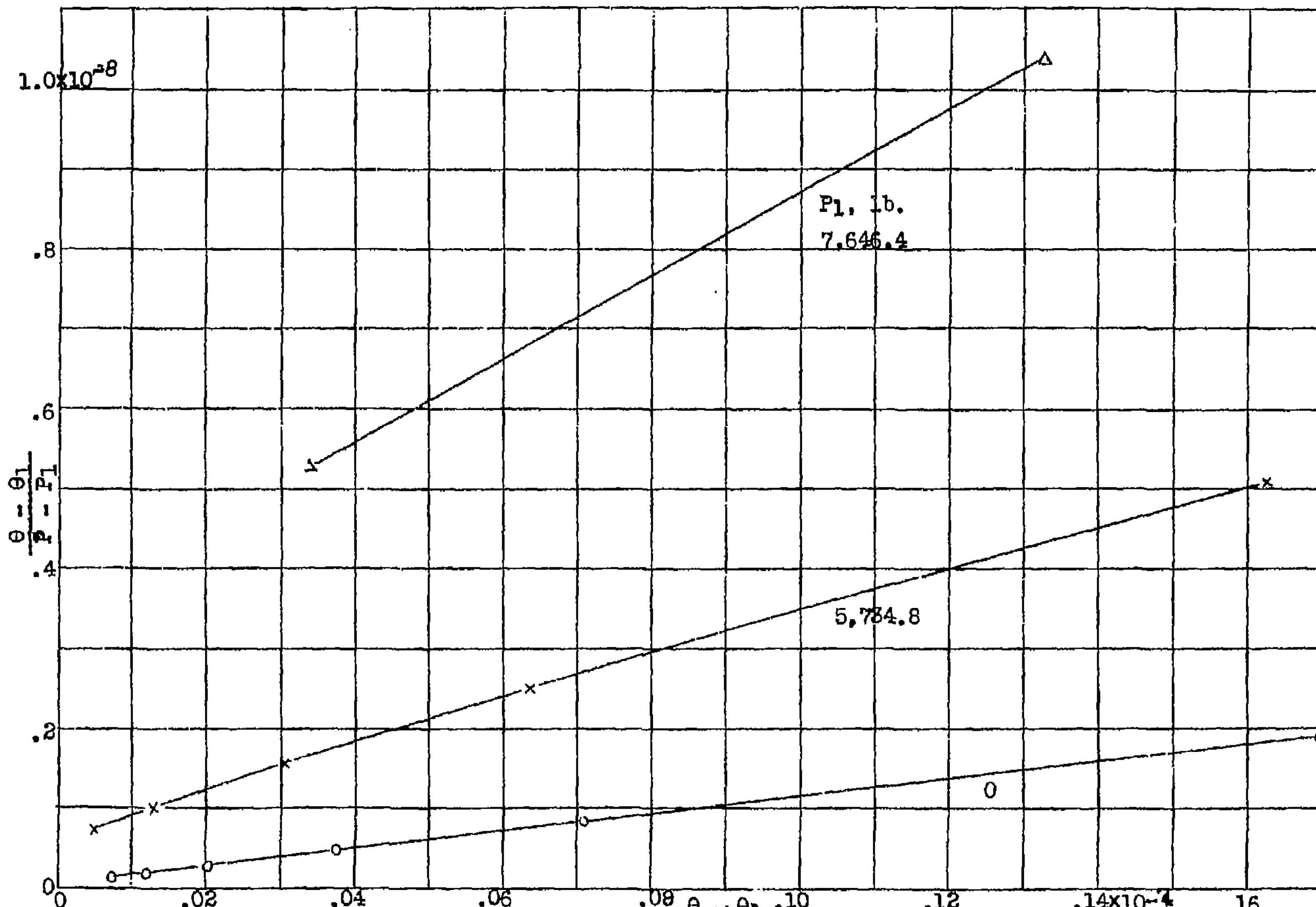

Figure 12.- Variation of $\frac{\theta - \theta_1}{P - P_1}$ with $\theta - \theta_1$ for problem 6.